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PT-symmetry in conventional quantum physics

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Abstract
Investigations during the last few years show that complex PT-symmetric or pseudo-Hermitian Hamiltonians possess a real discrete spectrum and several other features akin to a Hermitian Hamiltonian. These developments not only show us new directions but also demand a revisit to the conventional physics wherein PT-symmetry could have been invoked. After a review of some very interesting results, we present the super-barrier reflectionlessness and the cranking model of the nucleus. These two instances are being viewed as the meeting ground of the two: PT-symmetry and Hermiticity. The former presents an agreement and the latter is presently a dichotomy.

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1. Introduction

Relaxation of Hermiticity for the reality of the discrete spectrum has given rise to some very interesting investigations in the last few years [1]. The complex PT-symmetric Hamiltonians have been found to have a real discrete spectrum provided the energy eigenstates are also the eigenstates of PT; if not then the PT-symmetry is said to be spontaneously broken and there are complex-conjugate pairs of energy eigenvalues [2]. Here P stands for parity transformation ($x \to -x$) and T stands for time reversal ($\mathrm{i} \to -\mathrm{i}$).

The investigations regarding the possibilities with new kinds of Hamiltonians have been very exciting and fruitful. The question of the possibilities of bound states, energy band structure, transmission/reflection, real phase-space orbits akin to a real Hermitian Hamiltonian has been very important. In section 2, we review these possibilities and list the usual and unusual features of the new Hamiltonians with regard to the conventional quantum mechanics. In section 3, we review the pseudo-Hermiticity and random matrix theory. In section 4, we review the Gaussian pseudo-orthogonal and Gaussian pseudo-unitary ensembles G_pOE and G_pUE—new additions to the random matrix theory, when PT-symmetry is generalized in terms of pseudo-Hermiticity. In section 5, we bring out the PT-symmetric origin of super-barrier reflectionlessness of potentials. In section 6, the cranking model of the nucleus has been revisited to show that the complex energies of the Hamiltonian present a dichotomy as to whether the PT-symmetry is spontaneously broken or these are the resonances of a Hermitian Hamiltonian. We finally conclude in section 7.

2. Possibilities with complex PT-symmetric Hamiltonians

There has been a need to have an exactly solvable model which through a critical parameter can display both the scenarios of broken and unbroken PT-symmetries. This has been facilitated by the complex version of Scarf II potential; it has been shown that for the potential $V_S(x) = -V_1 \operatorname{sech}^2 x + \mathrm{i} V_2 \tanh x \operatorname{sech} x$, when $V_2 > V_1 + 1/4$, the eigenvalues make a transition from real to complex-conjugate pairs [3]. The new eigenstates are orthogonal in a new way, i.e. $\int \Psi_m^{PT}(x)\Psi_n(x)\,\mathrm{d}x = \epsilon_\pm \delta_{m,n}$. Mostly, one-dimensional potentials have been studied under PT-symmetry. An exactly solvable two-dimensional potential $V(r,\theta) = -\frac{\alpha}{r} + \frac{\mathrm{i}\beta\cos\theta}{r^2}$ has been solved in parabolic cylindrical coordinates to display the real discrete spectrum.

When complex PT-symmetric potentials become periodic, it is curious to know whether we will still have a real energy–momentum dispersion relation, $E(K)$ (energy band structure). Considering the additional Bloch condition on the wavefunctions, it turns out that $E(K)$ is real once again [4]; it however remains a conjecture only. It was reported that one gets an essentially unusual energy band structure. In the reduced Brillouin zone scheme, it was found that instead of sharp discontinuity at the zone boundary the $E(K)$ curve gets rounded in essentially leaving [4, 5] the boundary ($KL = \pi$). Equivalently, the antiperiodic wave solutions were found essentially missing for complex, PT-symmetric, periodic potentials.

By considering an interesting extension of the Kronig–Penny model $V_P(x) = (V_1 + \mathrm{i}V_2)\delta(x - b/2) + (V_1 - \mathrm{i}V_2)\delta(x - a - b/2)$ [6], it has been shown that one gets the usual band structure if $V_1 > V_2$. When V_2 becomes greater than V_1, one gets the unusual band structure where apart from some sharp band gaps one may also have rounded bands which may leave out both the boundaries $KL = 0, \pi$. This means that both periodic and antiperiodic wave solutions may or may not be missing.

When a potential is real it does not matter whether particles enter from the left or the right of the potential, the probability of reflection (transmission) does not show handedness. This is the consequence of time-reversal symmetry of the potential. The complex (optical, absorbing) potential breaks the time-reversal symmetry and it has been proved that when the total potential is not space symmetric the reflectivity shows handedness [7]. Thus, in general, reflection probability will depend on whether the particle enters from the left or from the right, if potential is complex and PT-symmetric. We find that if the real part of a complex PT-symmetric potential is positive definite the probabilities of reflection (R), transmission (T) and absorption (A) are normal (all less than unity), if the particle enters from an absorptive side. Otherwise, $R(E)$ at some energy or in a regime of energies becomes abnormal (>1) [8]. Complex PT-symmetric potential barriers, therefore, act like 'spy-glass' fitted in the window of a room. One can view outside without being viewed from outside. Some more interesting features of Scrödinger transmissions have been studied.

The concept of classical phase-space orbits has been a doorway from classical to semiclassical, quantal and statistical mechanics. Inspired by the success [2, 9] of the WKB method for a class of complex PT-symmetric potentials, it has been shown that complex PT-symmetric potentials admitting a real discrete spectrum too can have real phase-space orbits [10]. The phase space (x, p) gets segregated into two parts (x, p_r) and (x, p_i), the orbits in the former are symmetric and in the latter part they are anti-symmetric enclosing null area. However, remarkably the area enclosed by a symmetric orbit accounts correctly for the phase-space quantization *a la Sommerfeld*. The interesting nature of the turning points is very crucial here. These turning points are of the type $(-a + \mathrm{i}b, a + \mathrm{i}b)$; they follow from a more interesting property of the roots of a PT-symmetric equation which are always of the type $(-z^*, z)$. The existence of such a turning point is the *necessary* condition of finding a real discrete spectrum for a complex PT-symmetric potential. Very interestingly, the null spectrum of many PT-symmetric potentials can be explained by this criterion.

For potentials of the type $V(x) = -(\mathrm{i}x)^\nu$ [2], the semiclassical quantization suggests several branches of discrete spectra, say when $\nu > 5$. It would be interesting to study both exact quantal and semiclassical spectra and to explore a way of labelling them by a 'new quantum number'.

3. Pseudo-Hermiticity and random matrix theory

The Hamiltonian H is called η-pseudo-Hermitian if $\eta H \eta^{-1} = H$. That pseudo-Hermitian Hamiltonians may have a real discrete spectrum and the eigenstates are then pseudo-orthogonal as $\langle\psi_m|\eta\psi_n\rangle = \delta_{m,n}$ has been known for a long time [11]. However, after the knowledge of PT-symmetric quantum mechanics it has again been taken up more rigorously. It has been very remarkable [12] to suggest that PT-symmetry of a Hamiltonian is nothing but its P-pseudo-Hermiticity. Immediately after this, the puzzle of real discrete spectrum of non-PT-symmetric, complex Morse potential was resolved [13] by suggesting that this Hamiltonian is pseudo-Hermitian under $\eta = \mathrm{e}^{p\theta}$ (imaginary shift of the position coordinate). Also, several other non-PT-symmetric potentials were resolved to have a real discrete spectrum under $\eta = \mathrm{e}^{\mathrm{i}\phi(x)}$ ('gauge transformation') [14].

The concept of PT-symmetry has found a new mathematical framework in terms of pseudo-Hermiticity. However, since PT-symmetry is physically more appealing, it has been recasted [15, 17] in terms of pseudo-Hermiticity wherein the concepts of generalized parity, time reversal and charge conjugation have been evolved. Pseudo-Hermiticity is later discovered to be composed of ρ and μ [18] as $\eta = (\mu\rho^{-1})'$. Here, a Hamiltonian is termed as pseudo-real if $\rho H \rho^{-1} = H^*$ and pseudo-adjoint if $\mu H \mu^{-1} = H'$, where e.g. $\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)' = -\frac{\mathrm{d}}{\mathrm{d}x}$ and $H' = H^{\mathrm{Transpose}}$. The former yields a necessary condition for a spectrum to be real and the latter fixes the definition of the inner product for the eigenstates.

Studying pseudo-Hermiticity using matrices turns out to be very instructive in revealing many interesting properties of pseudo-Hermiticity in general [17, 18]. It is known that if H is Hermitian then $U = \mathrm{e}^{\mathrm{i}H}$ is unitary. It has been very interesting to realize that if G is p-H then $D = \mathrm{e}^{\mathrm{i}G}$ is pseudo-unitary and pseudo-unitarity is defined as $D^\dagger = \delta D^{-1}\delta^{-1}$ [19]. This development could give rise to the construction of new random matrix ensembles G_pUE and G_pOE.

Random matrix theory (RMT) [20, 21] was first proposed to study the nearest-neighbour level spacing distribution (NNLSD). Since level spacing requires two levels, in RMT a large ensemble of Hamiltonians (2×2 matrices) where the matrix elements come from a Gaussian population are studied. These matrices are symmetric, Hermitian and 4×4 symplectic. The first is invariant under time reversal, the second one is not and the third one is again invariant but includes Kramer's degeneracy. The spacing distributions yielded are written combinedly as $P_\beta(s) \sim s^\beta \mathrm{e}^{-s^2}$ with $\beta = 1, 2, 4$ [20, 21]. These are called Gaussian orthogonal (GO), Gaussian unitary (GU) and Gaussian symplectic (GS) spacing distribution functions, respectively. These ensembles are referred to as Wigner–Dyson ensembles. More interestingly, when $N \times N$ matrices were studied, the results were not appreciably different from $P_\beta(s)$ given above. The nuclear energy levels of the same J^π (angular momentum and parity) are found to obey $P_1(s)$ NNLDS. Most importantly, these known spacing distributions for small values of s show a tendency of linear, quadratic and quartic level repulsion, respectively. These characteristic features are used to decide whether the system is invariant under time reversal and other transformations.

4. New ensembles and spacing distribution: G_pOE and G_PUE

4.1. Pseudo-symmetric or complex symmetric matrix Hamiltonians

By Gaussian ensemble, we mean that the probability distribution of H is commonly given as

$$P(H) = \mathcal{N}\exp\left(-\frac{\mathrm{Tr}(HH^\dagger)}{2\sigma^2}\right). \tag{1}$$

Let us consider a matrix Hamiltonian H given below which is pseudo-Hermitian as $\eta H\eta^{-1} = H^\dagger$ and pseudo-real, i.e. $\rho H\rho^{-1} = H^*$ [18]. It is self-pseudo-adjoint or symmetric as $H' = H$. Here η are ρ are preferably involutory operators. It has got conditionally real eigenvalues iff $b^2 \geqslant c^2$. Here the prime, the asterisk ($\mathcal{K}_0$) and the dagger denote transpose, conjugate and transpose conjugate, respectively.

$$\begin{aligned} H &= \begin{bmatrix} a+b & \mathrm{i}c \\ \mathrm{i}c & a-b \end{bmatrix}, \qquad b^2 \geqslant c^2, \\ \eta = \rho &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad E_{1,2} = a \pm \sqrt{b^2 - c^2}. \end{aligned} \tag{2}$$

One can construct an antilinear commutant $\Theta = \rho^{-1}\mathcal{K}_0$ [16, 18] of H such that $[\Theta, H] = 0$ or $\Theta H\Theta^{-1} = H$ and $\Theta^2 = 1$. We would like to assert that here $P = \rho^{-1}$ and $T = \mathcal{K}_0$ and hence the antilinear symmetry $\Theta = PT$. When eigenvalues are real ($b^2 > c^2$), we have $PT\Psi_n = (-1)^n\Psi_n$. When $b^2 < c^2$, the PT-symmetry is spontaneously broken. This Hamiltonian in our opinion is another realization of Hamiltonians with antilinear symmetry as visualized by Haake (p 217 in [21]) as $[\mathcal{A}, \mathcal{D}] = 0$ such that $\mathcal{A}^2 = 1$.

We define pseudo-orthogonal transformation as ${}_pO' = \delta_p O^{-1}\delta^{-1}$ such that for any two arbitrary vectors from a linear space the scalar product remains invariant, i.e. $\tilde{x}'\delta\tilde{y} = x'\delta y$, where $\tilde{x} = {}_pOx$ and $\tilde{y} = {}_pOy$. Let us represent ${}_pO$, energy-eigenvalue matrix E and a metric δ

$${}_pO = \begin{bmatrix} \cosh\theta & \mathrm{i}\sinh\theta \\ -\mathrm{i}\sinh\theta & \cosh\theta \end{bmatrix}, \qquad \delta = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix} = \sigma_y, \qquad E = \begin{bmatrix} E_1 & 0 \\ 0 & E_2 \end{bmatrix}, \tag{3}$$

where $-\infty < \theta < \infty$. The single parameter matrix ${}_pO$, expressible as $\exp(2\mathrm{i}\theta J_2)$ with $J_2 = \frac{1}{2}\mathrm{i}\sigma_y$, constitutes a subgroup of $SU(1,1)$ [22]. A very important consequence of the group connection is that we can generate *all* possible H in (2) as $H = {}_pOE_pO^{-1}$. This provides us with a unique connection between (a, b, c) and (E_1, E_2, θ) and the consequent Jacobian is $\mathcal{J} = \frac{|s|}{8}$. We have $P(s) = \mathcal{N}sK_0\left(\frac{s^2}{2\sigma^2}\right)$. By finding $\langle s\rangle$ introducing $x = \frac{s}{\langle s\rangle}$, we eventually find the normalized NNLSD and call it as $P^{\mathrm{GPOE}}(x)$:

$$P^{\mathrm{GPOE}}(x) = \frac{\Gamma^4\left(-\frac{1}{4}\right)}{32\pi^3}xK_0\left(\frac{2\Gamma^4(3/4)}{\pi^2}x^2\right). \tag{4}$$

If we write it as $P^{\mathrm{GPOE}}(x) = \alpha xK_0(\beta x^2)$, we have $\alpha = 0.5818$ and $\beta = 0.4569$. When $0 < x < 0.5$, we have $P^{\mathrm{GPOE}}(x) \sim (0.5 - 1.2\ln x)x$. For any other pseudo-symmetric or complex symmetric matrix Hamiltonian that is composed of three independent Gaussian-random variables (a, b, c) appearing linearly in H, we claim that $P^{\mathrm{GPOE}}(x)$ is the universality. The new 'universality' shows a distinctly different behaviour as compared to the usual ones (see figure 1(*b*)).

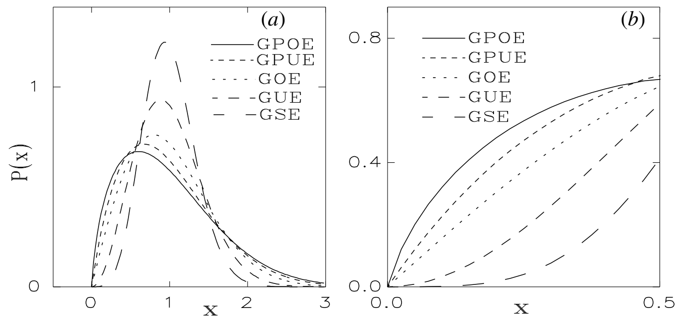

Figure 1. (*a*) Various spacing statistics, $P(x)$, see equations (4), (7) and the text; (*b*) $P(x)$ for $0 < x < 0.5$. Weaker level repulsion (higher $P(x)$ for small values of x) in the case of G_pOE and G_pUE than those of the known Wigner–Dyson distributions is believed to be the essence of PT-symmetric systems.

4.2. *Pseudo-Hermitian matrix Hamiltonians*

We now consider pseudo-Hermitian matrix Hamiltonians with four parameters (a, b, c, d)

$$H = \begin{bmatrix} a+b & d+\mathrm{i}c \\ -d+\mathrm{i}c & a-b \end{bmatrix}, \qquad e^2 = b^2 - c^2 + d^2 \geqslant 0,$$
$$\eta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad E_{1,2} = a \pm e. \tag{5}$$

Here we have $\eta H \eta^{-1} = H^\dagger$, P and T operators can be constructed as prescribed in [15, 17] and an antilinear commutant, Θ, of H can be constructed as prescribed in [16, 18]. Consider a transformation ${}_pU$ which preserves the pseudo-norm as $\tilde{x}^\dagger \eta \tilde{y} = x^\dagger \eta y$, where $\tilde{x} = {}_pUx$, $\tilde{y} = {}_pUy$. In doing so, ${}_pU$ would satisfy an interesting condition, i.e. ${}_pU^\dagger = \eta\, {}_pU^{-1} \eta^{-1}$ which is called pseudo-unitarity (see e.g. [19]).

A general three-parameter (θ, ψ, ϕ) matrix, ${}_pU$, which is pseudo-unitary under the same metric η (5), can be written as

$${}_pU = \begin{bmatrix} \mathrm{e}^{\mathrm{i}\psi}\cosh\theta & \mathrm{e}^{\mathrm{i}\phi}\sinh\theta \\ -\mathrm{e}^{-\mathrm{i}\phi}\sinh\theta & \mathrm{e}^{-\mathrm{i}\psi}\cosh\theta \end{bmatrix}, \qquad 0 \leqslant \phi, \psi \leqslant 2\pi, \quad 0 < \theta < \infty. \tag{6}$$

This constitutes a Lie group $SU(1, 1)$ [22] with generators as $J_0 = \frac{1}{4}\sigma_z$, $J_1 = \frac{1}{4}\mathrm{i}\sigma_y$, $J_3 = -\frac{1}{4}\mathrm{i}\sigma_x$. However, in order to construct the pseudo-Hermitian matrix (5) we require only two parameters in ${}_pU$. The same situation arises [20, 21] in the case of GUE, where two out of three parameters suffice in writing the unitary matrix U; nevertheless, it requires three parameters to have $SU(2)$. Thus, we take $\psi = 0$ in (6) and generate H in equation (5) as ${}_pUE_pU^{-1} = H$. This is how we go over to (E_1, E_2, ϕ, ψ) from (a, b, c, d). We find

$$P^{\mathrm{GPUE}}(x) = \frac{\mathcal{B}^2}{2(\sqrt{2}-1)} x \exp\left(\frac{\mathcal{B}^2 x^2}{4}\right) \mathrm{erfc}\left(\frac{\mathcal{B}x}{\sqrt{2}}\right), \qquad \mathcal{B} = \frac{2(\sqrt{2} - \log(1+\sqrt{2}))}{\sqrt{\pi}(\sqrt{2}-1)}. \tag{7}$$

If we write it as $P^{\mathrm{GPUE}}(x) = \alpha x\, \mathrm{e}^{\beta x^2} \mathrm{erfc}(\gamma x)$ where $\alpha = 2.5433$, $\beta = 0.5267$, $\gamma = 1.0263$, its linear dependence on x is deceptive and its behaviour near small values of x is actually curved (short-dashed line in figure 1(*b*)) lying below the curve corresponding to $P^{\mathrm{GPOE}}(x)$ (see the solid line in figure 1(*b*)). For $0 < x < 0.5$, we have $P^{\mathrm{GPUE}}(x) \sim 2.5x(1 - 0.95x)$.

The essence of pseudo-Hermiticity and hence that of PT-symmetry lies in the weaker level repulsion at smaller spacings as produced and shown here in figure 1. The enigmatic non-trivial zeros of the Riemann zeta function are found to follow GUE level spacing distribution. By constructing one more G_pUE, wherein the interaction is pseudo-Hermitian under a metric $\eta = \mathrm{diag}(\mathrm{e}^t, \mathrm{e}^{-t})$, we conclude that in the case of quasi-Hermiticity (when the metric is real, positive and diagonal) the spacing distribution for a domain of t remains hardly different from that of GUE. Consequently, based on this and other arguments, we speculate [24] that the most sought after Hamiltonian giving rise to enigmatic Riemann zeros could also be PT-symmetric.

5. Super-barrier reflectionlessness and PT-symmetry

5.1. A meeting ground of PT-symmetry and Hermiticity

Generally, the reflectivity $R(E)$ of the potential barriers such as $V_G(x) = V_0\,\mathrm{e}^{-x^2}$, $V_E(x) = V_0\,\mathrm{sech}^2 x$ and $V_L = \frac{V_0}{1+x^2}$ is a smooth decreasing function of energy [25] with a rectangular barrier as an exception. The rectangular potential entails reflectivity zeros above the barrier energies (super-barrier energies). Usually, the reflectivity zeros are passed off as artificial or arising from the interference between reflected waves from sharp edges. It is also found that if the above-mentioned barriers are truncated on both the sides the reflectivity may show oscillations [26]. The question as to whether there could be smooth potential barriers entailing reflectivity zeros/minima becomes very interesting. There are two very interesting examples in this regard. First, a very interesting semiclassical study of very weak reflection above the non-analyticity of a barrier reveals zeros in $R(E)$ for the potential $V_{\mathrm{Berry}}(x) = V_0(1 - \mathrm{e}^{1/|x|})$ [27]. Second, an inclusion [28] of two pairs of complex-conjugate turning points in the WKB approximation brings to the light the zeros in reflectivity for the smooth single piece potential $V_{\mathrm{Ch}}(x) = \frac{V_0}{1+x^4}$.

A recent study categorizes [29] the profiles $V_G(x)$, $V_L(x)$, $V_E(x)$ as type I. There are strictly two complex-conjugate turning points at energies above the barrier and a simple WKB method yields smoothly decreasing $R(E)$. It also very crucially points out that these barriers have top curvature as non-zero, i.e. $V''(0) = 0$, and the co-efficient of kurtosis as more than 3. In statistical [30] data analysis, the coefficient of kurtosis β_2 is a measure of flatness of a distribution function around its top. It is defined as

$$\beta_2 = \frac{\mu_4\mu_0}{\mu_2^2}, \qquad \mu_n = \int_{-\infty}^{\infty} (x-\bar{x})^n V(x)\,\mathrm{d}x, \qquad \bar{x} = \int_{-\infty}^{\infty} x V(x)\,\mathrm{d}x. \tag{8}$$

Type-II barriers are the profiles which have either top curvature as zero or β_2 lying in (1.8, 3.0), or both. The value 1.8 is the limiting (least) value of β_2 for any single-top profile and 3 is the value of β_2 for a Gaussian profile $V_G(x)$. Type II have been conjectured to possess reflectivity which has zeros/minima (see figure 2). Note that $V_{\mathrm{Berry}}(x)$ and $V_{\mathrm{Ch}}(x)$ are type II as per the condition of top curvature (their β_2 is indeterminate). Two examples of type-II barriers are $V_1(x,n) = V_0\,\mathrm{e}^{-x^{2n}}$ and $V_2(x,n) = \frac{V_0}{1+x^{2n}}$, $n = 2, 3$. The former has both the top curvature as zero and β_2 lying in (1.8, 3.0), and the latter profile has top curvature as zero and has finite β_2 only when $n \geqslant 3$.

In this section, we have covered hitherto known five criteria on a barrier to have reflectivity zeros. All of these could at best be sufficient but not necessary. Further studies indicate that a small deviation from symmetry of $V(x)$ leads to the disappearance of zeros or weakening of the minima. We thus believe that symmetry of the potential barriers is one of the necessary conditions (see figure 2).

Recently, we have shown [31] that the real discrete spectrum of the potentials of the type $V_K(x) = -x^{2K+2}$, $K = 1, 2, \ldots$, which are a sub-class of the well-known PT-symmetric

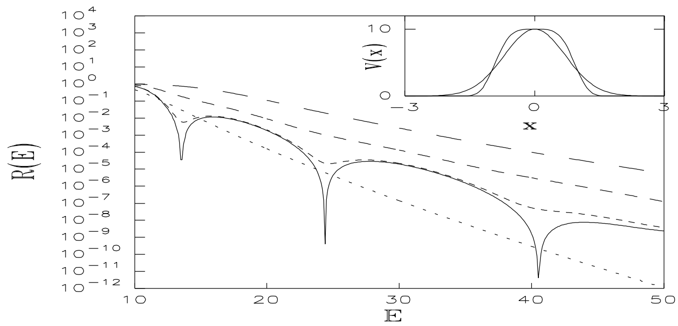

Figure 2. Exactly computed reflectivity, $R(E)$, for exponential potentials, $V_s(x) = V_0\,\mathrm{e}^{-x^4-sx}$. The dotted line displays the reflectivity for Gaussian barrier and the solid line is for $V_{s=0}(x)$. The short-dashed, medium-dashed and long-dashed lines display the reflectivity when asymmetry parameter s is 0.1, 0.5 and 1.0, respectively. Note that for the case of $s = 0.1$ feeble oscillations are still sustained. Here $V_0 = 10$ in arbitrary units. Note the flatness of the potential $V_{s=0}(x)$ (thick line) at the barrier top in comparison to the Gaussian barrier (thin line) in the inset. The value of the kurtosis parameter β_2 for the flat barrier is 2.18 ($s = 0$) and for Gaussian it is 3. The values of the kurtosis parameter, β_2, for the case of $s = 0.1, 0.5, 1.0$ are 2.46, 2.76 and 3.16, respectively. In the cases of $s = 0.5, 1.0$, an increased value of V_0 exhibits slightly more pronounced oscillations.

potential, i.e. $V_\nu(x) = -(\mathrm{i}x)^\nu$ [2], is nothing but the collection of reflectivity zeros of $V_K(x)$. Note that $V_K''(0) = 0$ but their β_2 is indeterminate. More crucially, we suggest that these discrete energies are the result of PT-symmetric boundary conditions on the eigenfunctions. Recall that this presents the scenario of PT-symmetry with a novelty wherein the eigenvalues are real and discrete and the eigenfunction is also an eigenstate of PT, yet we do no have bound states! The fact that eigenvalues of $V_\nu(x)$ as a function of ν are known to be smooth [2] and in agreement (without any discontinuity) say when $\nu = 4(K = 1)$ sets the stage for the meeting ground of PT-symmetry and Hermiticity.

We would like to present the PT-symmetric origin of reflectivity zeros in two more ways in the following. When the potential converges asymptotically, i.e. $V(\pm\infty) = 0$, the boundary conditions on the wavefunction, namely $\Psi(x) \sim \Theta(x)\,\mathrm{e}^{-kx} + \Theta(-x)\,\mathrm{e}^{kx}$, yield possible discrete eigenvalues of bound states. The boundary conditions on the wavefunction $\Psi(x) \sim \Theta(-x)\,\mathrm{e}^{-\mathrm{i}kx} + \Theta(x)\,\mathrm{e}^{\mathrm{i}kx}$ bring out the possible complex energy resonances. For reflectivity zeros, we require $\Psi(x) \sim \Theta(-x)\,\mathrm{e}^{\mathrm{i}kx} + \Theta(x)\,\mathrm{e}^{\mathrm{i}kx}$; note that $PT\Psi(x) = \Psi(x)$. Here $\Theta(x \geqslant 0) = 1, \Theta(x < 0) = 0$.

For exact quantal calculation of reflectivity zeros, we have to impose PT-symmetric boundary on the wavefunction. We show below that the same will be attained by PT-symmetric turning points in the semiclassical method. An extension of WKB has been suggested [28] wherein one can use two pairs of complex-conjugate turning points ($z_1 = -a + \mathrm{i}b, z_2 = a + \mathrm{i}b, z_1^* = -a - \mathrm{i}b, z_2^* = -a - \mathrm{i}b$) for $E > V_0$:

$$R(E) = |2\cos[\pi\alpha(E)]\,\mathrm{e}^{-\pi\beta(E)}|^2, \qquad \alpha(E) = \frac{1}{\pi}\int_{z_1}^{z_2} p(z)\,\mathrm{d}z, \qquad \beta(E) = \frac{1}{\pi}\int_{z_1^*}^{z_1} p(z)\,\mathrm{d}z. \tag{9}$$

We remark that z_1, z_2 are PT-symmetric turning points such that $z_1 = z, z_2 = -z^*$ and the condition of reflectionlessness in equation (9) becomes $\alpha(E) = (n + 1/2)$ which is nothing but the quantization law [10] for complex PT-symmetric potentials:

$$\frac{1}{\pi}\int_{z_1}^{z_2} p(z)\,\mathrm{d}z = (n + 1/2). \tag{10}$$

6. Complex eigenvalues in the cranking model of the nucleus

6.1. A dichotomy between PT-symmetry and Hermiticity

The exactly solvable Hamiltonian of the cranking model of the nucleus is due to Valatin [32] and is given as

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) + \frac{1}{2}m\omega_2^2(x^2 + y^2) + \frac{1}{2}m\omega_3^2 z^2 - \Omega J_x, \qquad \omega_3 < \omega_2. \tag{11}$$

The linear canonical transformations from y, z, p_y, p_z to new canonical Q_2, Q_3, P_2, P_3 are given as

$$\begin{aligned} &\alpha_2 Q_2 = \lambda_2[m(\alpha_2 - \Omega\beta_2)y - \beta_2 p_z], && m\alpha_3 Q_3 = \lambda_3[m(\alpha_3 + \Omega\beta_3)y - \beta_3 p_y],\\ &P_2 = \lambda_2[m(\alpha_2\beta_2 - \Omega)z + p_y], && P_3 = \lambda_3[m(\alpha_3\beta_3 + \Omega)z + p_z]. \end{aligned} \tag{12}$$

Various parameters are defined as

$$\begin{aligned} &\alpha_2^2 = \omega_+^2 + \Omega^2 + \sqrt{S}, \qquad \alpha_3^2 = \omega_+^2 + \Omega^2 - \sqrt{S}, \qquad \omega_+^2 = \frac{1}{2}\left(\omega_2^2 + \omega_3^2\right),\\ &\omega_-^2 = \frac{1}{2}\left(\omega_2^2 - \omega_3^2\right), \qquad S = (\omega_-^2)^2 + 4\Omega^2\omega_+^2, \qquad \beta_2 = 2\Omega\alpha_2\big/\left(\alpha_2^2 - \omega_3^2 + \Omega^2\right),\\ &\beta_3 = -2\Omega\alpha_3\big/\left(\alpha_3^2 - \omega_2^2 + \Omega^2\right), \qquad \lambda_2 = \sqrt{\frac{\alpha_2}{\mu_2}}, \qquad \lambda_3 = \sqrt{\frac{\alpha_3}{\mu_3}},\\ &\mu_2 = \alpha_2 + \alpha_3\beta_2\beta_3, \qquad \mu_3 = \alpha_3 + \alpha_2\beta_2\beta_3 \qquad \gamma = \frac{\mu_2}{\beta_2} = \frac{\mu_3}{\beta_3}. \end{aligned} \tag{13}$$

Several other interesting identities can be found in the appendix of [32]. The Hamiltonian H gets transformed to

$$H = \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_2^2 x^2 + \frac{1}{2m}P_2^2 + \frac{1}{2}m\alpha_2^2 Q_2^2 + \frac{1}{2m}P_3^2 + \frac{1}{2}m\alpha_3^2 Q_3^2. \tag{14}$$

When $H\Psi = E_{n_1,n_2,n_3}\psi$ and $\psi(x, y, z) = \phi(x)\chi_{n_2,n_3}(y, z)$, we get

$$E_{n_1,n_2,n_3} = (n_1 + 1/2)\hbar\omega_2 + (n_2 + 1/2)\hbar\alpha_2 + (n_3 + 1/2)\hbar\alpha_3. \tag{15}$$

The unnormalized eigenfunctions are

$$\begin{aligned} &\phi_{n_1}(x) = (p_x + \mathrm{i}m\omega x)^{n_1} \exp\left(-\frac{m\omega_2 x^2}{2\hbar}\right),\\ &\chi_{n_2,n_3}(y, z) = (P_2 + \mathrm{i}m\alpha_2 Q_2)^{n_2}(P_3 + \mathrm{i}m\alpha_3 Q_3)^{n_3}\chi_0(y, z),\\ &\chi_0(y, z) = \exp\left[-\frac{m}{2\hbar\tau}(\beta_2 y^2 + 2\mathrm{i}(1 - \sigma)yz + \beta_3 z^2)\right],\\ &\tau = (1 + \beta_2\beta_3)\gamma^{-1}, \qquad \sigma = \frac{\tau}{2}\left(\frac{\alpha_2}{\beta_2} + \frac{\alpha_3}{\beta_3}\right). \end{aligned} \tag{16}$$

Very interestingly, we find that the parameters β, τ, γ are odd functions of Ω, whereas the parameters α, μ, σ are even functions of Ω. Consequently, under time reversal, T, we have the following transformation laws:

$$\begin{aligned} &T\{\mathrm{i}, J_x, \Omega\}T^{-1} = -\{\mathrm{i}, J_x, \Omega\},\\ &T\{\beta, \tau, \gamma\}T^{-1} = -\{\beta, \tau, \gamma\},\\ &T\{\alpha, \mu, \sigma\}T^{-1} = +\{\alpha, \mu, \sigma\}. \end{aligned} \tag{17}$$

Under parity transformation, P, we have

$$\begin{aligned} &P\{x, y, z, p_x, p_y, p_z\}P^{-1} = -\{x, y, z, p_x, p_y, p_z\} \qquad \text{and}\\ &P\{\Omega, \mathrm{i}, J_x\}P^{-1} == +\{\Omega, \mathrm{i}, J_x\}. \end{aligned} \tag{18}$$

- The Hamiltonian H (11) would be ordinarily categorized as Hermitian, i.e. $H^\dagger = H$, using equations (7), (8), we claim that H is P-symmetric and T-symmetric to become PT-symmetric though trivially as
$$(PT)H(PT)^{-1} = H \qquad \text{and} \qquad PHP^{-1} = H = H^\dagger. \tag{19}$$
Let us rewrite the ground-state eigenfunction (6) as
$$\chi_{0,0}(y,z) = \exp[-(Ay^2 + Byz + Cz^2)], \qquad A = A_r + \mathrm{i}A_i, \\ B = B_r + \mathrm{i}B_i, \qquad C = C_r + \mathrm{i}C_i. \tag{20}$$
- One finds that when $\Omega < \omega_3$, the parameters $A, \mathrm{i}B, C$ are purely real and so are all the eigenvalues, $E_{n_1,n_2,n_3}(\Omega < \omega_3)$ (15). The eigenfunction $\chi_{0,0}(y,z)$ is real and vanishing as $y, z \to \pm\infty$, as we have $A_r > 0$ and the discriminant $\Delta(=B_r^2 - 4A_rC_r) < 0$. $\chi_{0,0}$ is also an eigenstate of PT as $PT\chi_{0,0}(y,z) = \chi_{0,0}(y,z)$ conforming to the exact symmetry scenario of a PT-symmetric Hamiltonian.
- When $\Omega > \omega_3$, we find that the energy eigenvalues (15) and the wavefunctions ($\chi_{0,0}(y,z)$) are real. However, $\chi_{0,0}(y,z)$ fails to satisfy the Dirichlet boundary condition as $A_r < 0$ and $\Delta > 0$. Thus, this regime is null for eigenvalues and needs to be excluded from (15). Let us remark that this regime is generally missed out.
- When $\omega_3 < \Omega < \omega_2$, the parameters $\alpha_2, \beta, \tau, \sigma$ turn out to be complex, all the eigenvalues (15) become complex (conjugate) and notably the eigenstates are *no more* the simultaneous eigenstates of PT. Very interestingly, in this regime $A_r > 0$ but the discriminant $\Delta = A_r^2 - 4B_rC_r$ becomes extremely small ($\sim 10^{-15}$) and a *spurious* function of Ω, its sign fluctuates and it is rendered *indefinite*.

Since in the regime (ω_3, ω_2) of Ω the eigenvalues are complex and $\Psi(x \sim \pm\infty, y \sim \pm\infty, z \sim \pm\infty) \sim 0$, the complex energies can represent resonances only when $\chi_{0,0}(y,z)$ diverges asymptotically. On the other hand, if $\chi_{0,0}(y,z)$ converges asymptotically, this regime would represent the spontaneous breaking of PT-symmetry. The convergence/divergence of $\chi_{0,0}(y,z)$ is decided solely by the sign of the discriminant, $\Delta(\Omega)$, and it being spurious gives rise to a dichotomy.

7. Conclusions

In conclusion, we wish that this paper succeeds in bringing out so many possibilities with new complex PT-symmetric Hamiltonians. Most interestingly, we get both usual and unusual results, the former are reassuring and the latter if observed may bring complex PT-symmetric interactions into contention. For instance, the unusual energy band structure in condensed matter physics or in wave propagation phenomenon may necessitate the use of interaction which breaks P and T individually yet preserves them jointly. Similarly, weaker level repulsion in the nearest-neighbour level spacing distribution for small spacings would indicate that the system is PT-symmetric. Several branches of spectra for a complex PT-symmetric potential as suggested by our phase-space quantization may bring out 'one more quantum number' to label even the spectrum of one-dimensional potential.

The instances such as super-barrier reflectionlessness and the complex eigenenergies of the Hamiltonian of the cranking model of the nucleus present a meeting ground for the two: PT-symmetry and Hermiticity. However, the latter instance remains an unresolved dichotomy.

Next, one may explore the possibility of working out statistical mechanics, Wigner distribution function, Freeman path integrals, Feynman diagrams and periodic orbit theory, etc with the complex PT-symmetric/pseudo-Hermitian Hamiltonians. Further, it would be worthwhile to enlist both the usual and unusual results in these studies.

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